SPECTRAL ORDER AND ISOTONIC DIFFERENTIAL OPERATORS OF LAGUERRE-PÓLYA TYPE

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ABSTRACT. The spectral order on $\mathbb{R}^n$ induces a natural partial ordering on the manifold $\mathcal{H}_n$ of monic hyperbolic polynomials of degree n. We show that all differential operators of Laguerre-Pólya type preserve the spectral order. We also establish a global monotony property for infinite families of deformations of these operators parametrized by the space l^∞ of real bounded sequences. As a consequence, we deduce that the monoid $\mathcal{A}'$ of linear operators that preserve averages of zero sets and hyperbolicity consists only of differential operators of Laguerre-Pólya type which are both extensive and isotonic. In particular, these results imply that any hyperbolic polynomial is the global minimum of its $\mathcal{A}'$-orbit and that Appell polynomials are characterized by a global minimum property with respect to the spectral order.

INTRODUCTION AND MAIN RESULTS

This is the third part of a series of papers [B, BBS, BP, BS] on the connections between linear operators acting on partially ordered manifolds of polynomials, the distribution of zeros of polynomials, and the theory of majorization.

Linear differential operators acting on various function spaces and classical majorization have both been extensively studied albeit so far only in separate contexts. On the one hand, differential operators of infinite order appear naturally in many applications. From a topological point of view they form a total set of linear continuous operators between spaces of differentiable functions [K], which is rather reminiscent of Peetre's abstract characterization of differential operators [P]. In this paper we are mainly concerned with linear operators of Laguerre-Pólya type, that is, infinite order differential operators induced by the Laguerre-Pólya class of entire functions. The significance of the latter stems from the fact that it consists precisely of those functions which are locally uniform limits in $\mathbb{C}$ of sequences of polynomials with all real zeros [L]. There is a very rich literature on differential operators of Laguerre-Pólya type and their applications to the study of the distribution of zeros of certain Fourier transforms, Pólya-Schoenberg frequency functions and totally positive matrices, the inversion and representation theories of convolution transforms, and the final set problem for trigonometric polynomials. Recently, such operators were also studied in connection with various generalizations of the Pólya-Wiman conjecture. Further details on these topics and related questions may be found in e.g. [CC1, CC2, KOW] and references therein.

On the other hand, the notion of (classical) majorization was first studied by economists early in the twentieth century as a means for altering the unevenness of distribution of wealth or income. Classical majorization was a key tool in Schur's work on Hadamard's determinantal inequality and the spectra of positive semidefinite Hermitian matrices [DK]. This notion was later formalized as a preorder on n-vectors of real numbers – also known as the spectral order on $\mathbb{R}^n$ – by Hardy,

Littlewood and Pólya in their study of symmetric means and analytic inequalities [HLP]. The spectral order has since found important applications in operator theory, convex analysis, combinatorics and statistics [An1, An2, MO]. As recent results have shown, classical majorization plays also a remarkable role in the study of quantum state mixing and efficient measurements in quantum mechanics [NV], quantum algorithm design [LMD] and the analysis of entanglement transformations in quantum computation and information theory [JP].

As we explain below, the spectral order on $\mathbb{R}^n$ induces a natural partial ordering $\preccurlyeq$ on the manifold $\mathcal{H}_n$ of monic univariate polynomials of degree n with all real zeros (cf. [B, BS]). Polynomials of this type are often called *hyperbolic* owing to the standard terminology used in the theory of partial differential equations [G], singularity theory and related topics [Ar]. Let $\Pi := \mathbb{C}[x]$ be the space of complex univariate polynomials regarded as functions on the complex plane. The main purpose of this paper is to study the properties of the posets $(\mathcal{H}_n, \preccurlyeq)$, $n \in \mathbb{N}$, under the action of hyperbolicity-preserving linear operators, that is, operators acting on Π that map hyperbolic polynomials to hyperbolic polynomials. Given a monic polynomial $P \in \Pi$ with $\deg P = n \geq 1$ we define $\mathcal{Z}(P)$ to be the unordered n-tuple consisting of the zeros of P, each zero occurring as many times as its multiplicity. Thus $\mathcal{Z}(P) \in \mathbb{C}^n/\Sigma_n$, where Σ_n is the symmetric group on n elements. We denote by $\Re\mathcal{Z}(P)$ the unordered n-tuple whose components are the real parts of the points in $\mathcal{Z}(P)$. Note that P is hyperbolic if and only if $\Re\mathcal{Z}(P) = \mathcal{Z}(P)$. A hyperbolic polynomial with simple zeros is called *strictly hyperbolic*. Let $\mathcal{H}_n \subset \Pi$ be the real manifold of monic hyperbolic polynomials of degree n. We extend this notation to $n = 0$ by setting $\mathcal{H}_0 = \{1\} \subset \Pi$. Clearly, for $n \geq 1$ one has a natural set-theoretic identification between $\mathcal{H}_n$ and $\mathbb{R}^n/\Sigma_n$ by means of the *root map*

$$\mathcal{Z} : \mathcal{H}_n \longrightarrow \mathbb{R}^n/\Sigma_n$$
$$P \longmapsto \mathcal{Z}(P). \tag{0.1}$$

The following theorem is due to Hardy, Littlewood and Pólya [HLP]:

Theorem 1. *Let $X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n/\Sigma_n$, $Y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n/\Sigma_n$. The following conditions are equivalent:*

  (i) *For any convex function $f : \mathbb{R} \to \mathbb{R}$ one has $\sum_{i=1}^n f(x_i) \leq \sum_{i=1}^n f(y_i)$.*

  (ii) *There exists a doubly stochastic $n \times n$ matrix A such that $\tilde{X} = A\tilde{Y}$, where $\tilde{X}$ and $\tilde{Y}$ are column n-vectors obtained by some (and then any) ordering of the components of X and Y, respectively.*

Theorem 1 defines what is usually known as classical majorization or the spectral order on $\mathbb{R}^n$: if the conditions of the theorem are satisfied we say that X is *majorized* by Y or that X is *less than Y in the spectral order*, which we denote by $X \prec Y$. One can easily check that if $X \prec Y$ then $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Note that although the spectral order is only a preordering on $\mathbb{R}^n$, Birkhoff's theorem [MO, Theorem 2.A.2] implies that it actually induces a partial ordering on $\mathbb{R}^n/\Sigma_n$. Therefore, Theorem 1 allows us to define a poset structure $(\mathcal{H}_n, \preccurlyeq)$ by setting $Q \preccurlyeq P$ whenever $P, Q \in \mathcal{H}_n$ and $\mathcal{Z}(Q) \prec \mathcal{Z}(P)$. In this way we may view the spectral order on $\mathbb{R}^n$ as a natural partial ordering on the manifold $\mathcal{H}_n$, which we call the *spectral order on $\mathcal{H}_n$*.

We can now state the following isotonicity theorem, which is our first main result:

Theorem 2. *Let $n \geq 1$ and $P, Q \in \mathcal{H}_n$ be such that $Q \preccurlyeq P$. Then for any $\lambda \in \mathbb{R}$ one has $Q - \lambda Q' \preccurlyeq P - \lambda P'$.*

This has several natural consequences. Recall that by a classical result of Pólya all differential operators of Laguerre-Pólya type are hyperbolicity-preserving, see e.g. [RS, Theorem 5.4.13]. Theorem 2 implies that much more is actually true,

namely all such operators preserve in fact the spectral order (Corollary 1). In particular, any degree-preserving differential operator of Laguerre-Pólya type is isotonic with respect to the partial ordering $\preccurlyeq$ on the manifold $\mathcal{H}_n$ for all $n \in \mathbb{N}$ (Corollary 2). This gives a new characterization of the sequence of Appell polynomials associated with an arbitrary function in the Laguerre-Pólya class by means of a global minimum property with respect to the spectral order (Corollary 3).

Let $D = d/dx$ denote differentiation with respect to x. The second main result of this paper is the following monotonicity theorem:

Theorem 3. *Fix $n \geq 1$ and let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1 \lambda_2 \geq 0$ and $|\lambda_1| \leq |\lambda_2|$. Then $(1 - \lambda_1 D)e^{\lambda_1 D}P \preccurlyeq (1 - \lambda_2 D)e^{\lambda_2 D}P$ for any $P \in \mathcal{H}_n$.*

Theorem 3 allows us to study the orbit of an arbitrarily given hyperbolic polynomial under the action of the monoid of differential operators of Laguerre-Pólya type. We equip the space l^∞ of real bounded sequences with a natural partial ordering $\leqslant$ and define infinite families of deformations of differential operators of Laguerre-Pólya type which are parametrized by vectors in l^∞. From Theorem 3 we deduce that any such family satisfies a global monotony property with respect to both partial orderings $\preccurlyeq$ and $\leqslant$ (Corollary 5). Moreover, these partial orderings are compatible with each other (Corollary 6). It follows that the monoid $\mathcal{A}'$ of all linear operators that act on each of the manifolds $\mathcal{H}_n$, $n \geq 1$, and preserve averages of zero sets consists only of differential operators of Laguerre-Pólya type which are extensive with respect to $\preccurlyeq$ (Corollary 7). Thus, any hyperbolic polynomial is the global minimum of its $\mathcal{A}'$-orbit with respect to the spectral order (Corollary 8).

The above results have further applications to the distribution of zeros of hyperbolic polynomials under the action of differential operators of Laguerre-Pólya type (Corollaries 9-11). At the same time, they seem to suggest even deeper connections between linear (differential) operators, the distribution of zeros of real entire functions, and the theory of majorization. As we point out in §3, it would be interesting to know whether appropriate modifications of the aforementioned results could hold for transcendental entire functions in the Laguerre-Pólya class. On the other hand, these results and those of [B, BP, BS] hint at the possible existence of an "analytic theory of classical majorization" and may therefore also be seen as natural steps towards developing such a theory. Problem 2 in [B] and Problems 1-3 in §3 are intended as further steps in this direction.

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1. Theorem 2 and applications

1.1. **Proof of Theorem 2.** A key ingredient in the proofs of Theorems 2 and 3 is the following criterion due to Hardy, Littlewood and Pólya [HLP].

Theorem 4. *Let $X = (x_1 \leq x_2 \leq \ldots \leq x_n)$ and $Y = (y_1 \leq y_2 \leq \ldots \leq y_n)$ be two n-tuples of real numbers. Then $X \prec Y$ if and only if the x_i's and the y_i's satisfy the following conditions:*

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \ \text{ and } \ \sum_{i=0}^{k} x_{n-i} \leq \sum_{i=0}^{k} y_{n-i} \text{ for } 0 \leq k \leq n-2.$$

We also make extensive use of *contractions*, a special kind of degree-preserving transformations acting on hyperbolic polynomials that we define as follows.

Definition 1. Let $P(x) = \prod_{i=1}^{n}(x - x_i) \in \mathcal{H}_n$, $n \geq 2$, and $1 \leq k < l \leq n$. Assume that $x_i \leq x_{i+1}$, $1 \leq i \leq n-1$, and that $x_k \neq x_l$. Let further $t \in \left(0, \frac{x_l - x_k}{2}\right]$ and define $Q \in \mathcal{H}_n$ to be the polynomial with zeros y_i, $1 \leq i \leq n$, where $y_k = x_k + t$,

$y_l = x_l - t$, and $y_i = x_i$, $i \neq k, l$. The polynomial Q is called the *contraction of P of type (k,l) and coefficient t* and is denoted by $Q = \mathcal{T}(k,l;t)P$. The contraction $\mathcal{T}(k,l;t)$ of P is called *simple* if $l = k+1$ and it is called *nondegenerate* if $t \neq \frac{x_l - x_k}{2}$.

Remark 1. The simple nondegenerate contractions in Definition 1 may be viewed as elementary versions of the so-called T-transforms for n-tuples of real numbers. The latter are essentially a mathematical formulation of Dalton's "principle of transfers" (see [MO]) and were first used by Hardy, Littlewood and Pólya in [HLP].

The proof of Theorem 2 builds on several auxiliary technical results. The first two of these, Proposition 1 and Lemma 1 below, may also be restated in terms of n-tuples of real numbers or doubly stochastic matrices in view of Theorems 1 and 4. However, since we are interested in the dynamics of polynomial zeros under the action of certain operators, it is convenient to formulate all the results exclusively in terms of polynomials.

Proposition 1. *Let $P, Q \in \mathcal{H}_n$ be two distinct strictly hyperbolic polynomials such that $Q \preccurlyeq P$. Then there exists a finite sequence of strictly hyperbolic polynomials $P_1, \ldots, P_m \in \mathcal{H}_n$ such that $P_1 = P$, $P_m = Q$ and P_{i+1} is a simple nondegenerate contraction of P_i for $1 \leq i \leq m-1$.*

The algorithm described in the next lemma will be used to give a constructive proof of Proposition 1.

Lemma 1. *Let $a < b$, $\sigma \in \left(0, \frac{b-a}{2}\right)$ and $p \in \mathbb{N}$. Assume that z_i, $1 \leq i \leq p$, are real numbers that satisfy $a + \sigma < z_1 < \ldots < z_p < b - \sigma$ and set*

$$P(x) = (x-a)(x-b)\prod_{i=1}^{p}(x - z_i), \ Q(x) = (x-a-\sigma)(x-b+\sigma)\prod_{i=1}^{p}(x - z_i).$$

There exist simple nondegenerate contractions $\mathcal{T}_1, \ldots, \mathcal{T}_s$ such that $Q = \mathcal{T}_s \cdots \mathcal{T}_1 P$.

Proof. Set $x_1 = a$, $x_{p+2} = b$ and $x_i = z_{i-1}$, $2 \leq i \leq p+1$, so that we may write

$$P(x) = \prod_{i=1}^{p+2}(x - x_i) \text{ with } x_i < x_{i+1}, \ 1 \leq i \leq p+1.$$

Choose $d \in \mathbb{N}$ such that $\sigma < 2^{d-1}\min(z_1 - a - \sigma, \ b - z_p - \sigma)$ if $p = 1$ and

$$\sigma < 2^{d-1}\min\left(z_1 - a - \sigma, \ b - z_p - \sigma, \ \min_{1 \leq i \leq p-1}(z_{i+1} - z_i)\right) \text{ if } p \geq 2.$$

We let $t = \frac{\sigma}{2^d}$ and build a finite sequence of polynomials $\{S_{1,i}\}_{i=0}^{p+1}$ as follows:

$$S_{1,0} = P \text{ and } S_{1,i} = \mathcal{T}(i, i+1; t)S_{1,i-1}, \ 1 \leq i \leq p+1.$$

Clearly, the contractions used in constructing this sequence are all simple. These contractions are also nondegenerate since

$$x_{i+1} - (x_i - t) > 2t, \ 1 \leq i \leq p+1,$$

by the choice of t. Thus, all polynomials $S_{1,i}$, $0 \leq i \leq p+1$, are strictly hyperbolic. In particular, this is true for the polynomial

$$P_1(x) := S_{1,p+1}(x) = \prod_{i=1}^{p+2}\left(x - x_i^{(1)}\right),$$

where $x_1^{(1)} = a + t$, $x_{p+2}^{(1)} = b - t$ and $x_i^{(1)} = z_{i-1}$, $2 \leq i \leq p+1$, so that $x_i^{(1)} < x_{i+1}^{(1)}$ for $1 \leq i \leq p+1$. We now use the same contractions as above to construct a finite sequence of polynomials $\{S_{2,i}\}_{i=0}^{p+1}$ starting with the polynomial P_1:

$$S_{2,0} = P_1 \text{ and } S_{2,i} = \mathcal{T}(i, i+1; t)S_{2,i-1}, \ 1 \leq i \leq p+1.$$

Repeating this procedure r times we arrive at the polynomial

$$P_r(x) := S_{r,p+1}(x) = \prod_{i=1}^{p+2} \left(x - x_i^{(r)} \right),$$

where $x_1^{(r)} = a + rt$, $x_{p+2}^{(r)} = b - rt$ and $x_i^{(r)} = z_{i-1}$ for $2 \le i \le p+1$. It is clear that all the contractions used in constructing the polynomial P_r are simple. Moreover, one can easily check that if $r \le 2^d$ then

$$x_{i+1}^{(r)} - \left(x_i^{(r)} - t \right) > 2t, \ 1 \le i \le p+1.$$

Since $Q = P_{2^d}$ the above algorithm shows that Q may be constructed from P by using a total of $s = (p+1)2^d$ simple nondegenerate contractions. $\qquad\square$

Definition 2. Let $P(x) = \prod_{i=1}^n (x - x_i)$ and $Q(x) = \prod_{i=1}^n (x - y_i)$ be two hyperbolic polynomials of degree $n \ge 1$ whose zeros are arranged in nondecreasing order, so that if $n \ge 2$ then $x_i \le x_{i+1}$ and $y_i \le y_{i+1}$ for $1 \le i \le n - 1$. The number

$$\delta(P, Q) := \sharp \left\{ i \in \{1, \dots, n\} \mid x_i \neq y_i \right\}$$

is called the *discrepancy* between P and Q.

Remark 2. It is clear from Definition 2 that $P = Q$ if and only if $\delta(P, Q) = 0$.

Proof of Proposition 1. The proposition is clearly true if $n = 2$ and we may therefore assume that $n \ge 3$. Let $x_1 < x_2 < \dots < x_n$ and $y_1 < y_2 < \dots < y_n$ denote the zeros of P and Q, respectively. Let further $r = \delta(P, Q)$ and note that $r \ge 1$ since P and Q are distinct polynomials. Actually, since the condition $Q \preccurlyeq P$ implies that $\sum_{i=1}^n y_i = \sum_{i=1}^n x_i$ we see that $r \ge 2$. We now prove the proposition by induction on r. If $r = 2$ then by Theorem 4 there exist indices $1 \le i < j \le n$ such that $y_k = x_k$ whenever $k \neq i, j$ and $y_i = x_i + \sigma$ while $y_j = x_j - \sigma$ for some $\sigma \in \mathbb{R}$ that satisfies

$$0 < \sigma < \min \left(x_{i+1} - x_i, x_j - x_{j-1}, \frac{x_j - x_i}{2} \right).$$

This means that if $j = i+1$ then Q is already a simple nondegenerate contraction of P. If this is not the case then Lemma 1 implies that Q may be obtained from P by the successive application of a finite number of simple nondegenerate contractions, which proves the result for $r = 2$.

Suppose that $r \ge 3$ and assume that the proposition is true for all pairs of strictly hyperbolic polynomials whose discrepancies are at most $r-1$. Since $\sum_{i=1}^n (x_i - y_i) = 0$ there must exist both positive and negative numbers among the differences $x_i - y_i$, $1 \le i \le n$. A close examination of consecutive differences shows that at least one the following cases has to occur:

Case 1. There exists $i \in \{1, 2, \dots, n\}$ such that $x_i < y_i$ and $x_{i+1} > y_{i+1}$. Define the polynomial $R = \mathcal{T}(i, i+1; t)P \in \mathcal{H}_n$, where $t = \min(y_i - x_i, x_{i+1} - y_{i+1})$. Note that $t \in \left(0, \frac{x_{i+1} - x_i}{2} \right)$ and thus R is a simple nondegenerate contraction of P. We now use Theorem 4 to check that one also has $Q \preccurlyeq R$. This is obvious if $i = 1$ and we may therefore assume that $i \ge 2$. It is then clear that

$$\sum_{k=1}^m x_k \le \sum_{k=1}^m y_k \ \text{if} \ m \le i - 1 \ \text{and} \ \sum_{k=m}^n x_k \ge \sum_{k=m}^n y_k \ \text{if} \ m \ge i + 2.$$

Moreover, using the fact that $Q \preccurlyeq P$ we get

$$(x_i + t) + \sum_{k=1}^{i-1} x_k = y_i + \sum_{k=1}^{i-1} x_k \le \sum_{k=1}^{i} y_k \text{ and}$$

$$(x_{i+1} - t) + (x_i + t) + \sum_{k=1}^{i-1} x_k = \sum_{k=1}^{i+1} x_k \le \sum_{k=1}^{i+1} y_k,$$

which shows that if $t = y_i - x_i$ then the zeros of Q and R satisfy the inequalities in Theorem 4. It follows that R is a strictly hyperbolic polynomials that satisfies $Q \preccurlyeq R$ and $\delta(Q, R) \le r - 1$. Similar computations show that these relations remain true if $t = x_{i+1} - y_{i+1}$. By assumption, Q may be obtained from R by the successive application of a finite number of simple nondegenerate contractions. Since R itself is a simple nondegenerate contraction of P, this proves the proposition in this case.

Case 2. There exist indices $i, j \in \{1, 2, \ldots, n\}$ with $j \ge i + 2$ such that $x_i < y_i$, $x_j > y_j$ and $x_k = y_k$ for $i + 1 \le k \le j - 1$. Let $\sigma = \min(y_i - x_i, x_j - y_j)$ and set

$$R(x) := (x - x_i - \sigma)(x - x_j + \sigma) \prod_{\substack{k=1 \\ k \ne i, j}}^{n} (x - x_k),$$

so that R is a strictly hyperbolic polynomial that satisfies $R \preccurlyeq P$. Note that since $\sigma \in \left(0, \frac{x_j - x_i}{2} \right)$ it follows from Lemma 1 that R may be constructed by applying to P a finite number of simple nondegenerate contractions. Clearly, these contractions affect only the zeros of P and its successive transforms that lie in the interval $[x_i, x_j]$. Computations similar to those used in case 1 show that $Q \preccurlyeq R$. Moreover, it is clear that $\delta(Q, R) \le r - 1$. Using again the induction assumption we deduce that Q may be obtained from R and therefore also from P by the successive application of a finite number of simple nondegenerate contractions, which completes the proof. $\square$

Before proceeding with the proof of Theorem 2 let us point out that if the non-degeneracy condition is omitted then minor modifications of the above arguments yield an analog of Proposition 1 for polynomials with multiple zeros. This result will not be used in the sequel and so we state it without proof:

Proposition 2. *Let P and Q be distinct polynomials in $\mathcal{H}_n$ that satisfy $Q \preccurlyeq P$. There exists a finite sequence of hyperbolic polynomials $P_1, \ldots, P_m \in \mathcal{H}_n$ such that $P_1 = P$, $P_m = Q$ and P_{i+1} is a simple contraction of P_i for $1 \le i \le m - 1$.* $\square$

The following proposition is the main step in the proof of Theorem 2.

Proposition 3. *If P and Q are strictly hyperbolic polynomials in $\mathcal{H}_n$ such that Q is a simple nondegenerate contraction of P then $Q - \lambda Q' \preccurlyeq P - \lambda P'$ for any $\lambda \in \mathbb{R}$.*

For the proof of Proposition 3 we need several additional results. Let us first fix the notation that we shall use throughout this proof.

Notation 1. We start with a strictly hyperbolic polynomial $P \in \mathcal{H}_n$ given by

$$P(x) = \prod_{i=1}^{n} (x - x_i) \text{ and } P'(x) = n \prod_{j=1}^{n-1} (x - w_j).$$

By Rolle's theorem we may label the zeros of P and P' so that

$$x_1 < w_1 < x_2 < \ldots < x_{n-1} < w_{n-1} < x_n,$$

which we assume henceforth. In most of the arguments below we shall also tacitly assume that $n \ge 3$. Fix an index $i \in \{1, 2, \ldots, n - 1\}$ and set $I = \left(0, \frac{x_{i+1} - x_i}{2} \right)$. For

$t \in \bar{I}$ we let $P_t \in \mathcal{H}_n$ denote the polynomial

$$P_t(x) = (x - x_i - t)(x - x_{i+1} + t) \prod_{\substack{k=1 \\ k \neq i, i+1}}^{n} (x - x_k)$$

and we define the following homotopy of polynomial pencils:

$$P(\lambda, t; x) = P_t(x) - \lambda P_t'(x), \text{ where } (\lambda, t) \in \mathbb{R} \times \bar{I} \text{ and } P_t'(x) = \frac{\partial}{\partial x} P_t(x).$$

Note that P_t is a strictly hyperbolic polynomial whenever $t \in \{0\} \cup I$ and so by the Hermite-Poulain-Jensen theorem [RS, Theorem 5.4.9] the polynomial $P(\lambda, t; x)$ is strictly hyperbolic for all $(\lambda, t) \in \mathbb{R} \times (\{0\} \cup I)$. Actually, if $0 < \varepsilon < \min(x_i - x_{i-1}, x_{i+2} - x_{i+1})$ then the same arguments show that the polynomial $P(\lambda, t; x)$ has only simple (real) zeros for any $(\lambda, t) \in \mathbb{R} \times \left(-\varepsilon, \frac{x_{i+1} - x_i}{2}\right)$. If we now fix such an ε it follows from the implicit function theorem that the zeros of $P(\lambda, t; x)$ are real analytic functions of (λ, t) in the domain $\mathbb{R} \times \left(-\varepsilon, \frac{x_{i+1} - x_i}{2}\right)$. Therefore, if we write

$$P(\lambda, t; x) = \prod_{k=1}^{n} (x - x_k(\lambda, t)) \text{ and } P'(\lambda, t; x) := \frac{\partial}{\partial x} P(\lambda, t; x) = n \prod_{l=1}^{n-1} (x - w_l(\lambda, t))$$

and further assume that the zeros and the critical points of $P(\lambda, t; x)$ are labeled so that $x_k(0, 0) = x_k$, $1 \leq k \leq n$, and $w_l(0, 0) = w_l$, $1 \leq l \leq n - 1$, then one has

$$x_1(\lambda, t) < w_1(\lambda, t) < x_2(\lambda, t) < \ldots < x_{n-1}(\lambda, t) < w_{n-1}(\lambda, t) < x_n(\lambda, t) \quad (1.1)$$

if $(\lambda, t) \in \mathbb{R} \times (\{0\} \cup I)$. These notations will be used in all lemmas below.

Lemma 2. *If $1 \leq k \leq n$ and $(\lambda, t) \in \mathbb{R} \times (\{0\} \cup I)$ then $P'(\lambda, t; x_k(\lambda, t)) \neq 0$ and*

$$\frac{\partial}{\partial \lambda} x_k(\lambda, t) = \frac{P_t'(x_k(\lambda, t))}{P'(\lambda, t; x_k(\lambda, t))} > 0.$$

In particular, for all $j \in \{1, 2, \ldots, n - 1\}$ one has

$$x_j(\lambda, t) < w_j(0, t) < x_{j+1}(\lambda, t) \text{ and } \lim_{\lambda \to \infty} x_j(\lambda, t) = \lim_{\lambda \to -\infty} x_{j+1}(\lambda, t) = w_j(0, t).$$

Moreover, $\lim_{\lambda \to \infty} x_n(\lambda, t) = -\lim_{\lambda \to -\infty} x_1(\lambda, t) = \infty$.

Proof. The first assertion follows from the fact that $P(\lambda, t; x)$ is strictly hyperbolic and $P(\lambda, t; x_k(\lambda, t)) = 0$. Implicit differentiation with respect to λ in the identity

$$P_t(x_k(\lambda, t)) - \lambda P_t'(x_k(\lambda, t)) = 0$$

yields immediately the equality stated in the lemma. Note that since P_t is strictly hyperbolic we have $P_t'(x_k(\lambda, t)) \neq 0$, so that if we let $P_t''(x) = \frac{\partial}{\partial x} P_t'(x)$ then

$$[P_t'(x_k(\lambda, t))]^2 \left[\frac{\partial}{\partial \lambda} x_k(\lambda, t)\right]^{-1} = [P_t'(x_k(\lambda, t))]^2 - P_t(x_k(\lambda, t))P_t''(x_k(\lambda, t)) > 0$$

by Laguerre's inequality for (strictly) hyperbolic polynomials [RS, Lemma 5.4.4]. If $t \in \{0\} \cup I$ is fixed then $-\lambda^{-1} P(\lambda, t; x) \to P_t'(x)$ as $|\lambda| \to \infty$ uniformly on compact sets. It follows that for $1 \leq j \leq n - 1$ one has $x_j(\lambda, t) < \lim_{\mu \to \infty} x_j(\mu, t) = w_j(0, t)$ and $x_{j+1}(\lambda, t) > \lim_{\mu \to -\infty} x_{j+1}(\mu, t) = w_j(0, t)$, which finishes the proof. $\square$

For $1 \leq k \leq n$ and $(\lambda, t) \in \mathbb{R} \times (\{0\} \cup I)$ we define the following expressions:

$$F_k(\lambda, t) = \left[\frac{P_t(x_k(\lambda, t))}{(x_k(\lambda, t) - x_i - t)(x_k(\lambda, t) - x_{i+1} + t)P_t'(x_k(\lambda, t))}\right]^2 \text{ if } \lambda \neq 0,$$

$$F_i(0, t) = F_{i+1}(0, t) = \frac{1}{(2t + x_i - x_{i+1})^2} \text{ and } F_k(0, t) = 0 \text{ if } k \neq i, i + 1.$$

(1.2)

Note that $F_k(0, t) = \lim_{\lambda \to 0} F_k(\lambda, t)$ for all $k \in \{1, 2, \ldots, n\}$ and $t \in \{0\} \cup I$.

Lemma 3. *If $1 \leq k \leq n$ and $(\lambda, t) \in \mathbb{R} \times (\{0\} \cup I)$ then*

$$\frac{\partial}{\partial t} x_k(\lambda, t) = (2x_k(\lambda, t) - x_i - x_{i+1})(2t + x_i - x_{i+1}) F_k(\lambda, t) \frac{\partial}{\partial \lambda} x_k(\lambda, t),$$

where $F_k(\lambda, t)$ is as in (1.2).

Proof. By Lemma 2 one has $\frac{\partial}{\partial \lambda} x_k(\lambda, t)\big|_{(0,t)} = 1$ for all $t \in \{0\} \cup I$ and $1 \leq k \leq n$. Moreover, it is clear that $\frac{\partial}{\partial t} x_k(\lambda, t)\big|_{(0,t)} = 0$ if $k \neq i, i+1$ while $\frac{\partial}{\partial t} x_i(\lambda, t)\big|_{(0,t)} = -\frac{\partial}{\partial t} x_{i+1}(\lambda, t)\big|_{(0,t)} = 1$. Thus, if $\lambda = 0$ then the lemma is a consequence of (1.2).

Assume now that $\lambda \neq 0$, so that

$$\frac{1}{\lambda} = \frac{P_t'(x_k(\lambda, t))}{P_t(x_k(\lambda, t))} = \frac{1}{x_k(\lambda, t) - x_i - t} + \frac{1}{x_k(\lambda, t) - x_{i+1} + t} + \sum_{\substack{r=1 \\ r \neq i, i+1}}^{n} \frac{1}{x_k(\lambda, t) - x_r}.$$

Applying $\frac{\partial}{\partial t}$ to the relation $P_t(x_k(\lambda, t)) - \lambda P_t'(x_k(\lambda, t)) = 0$ we get

$$[P_t'(x_k(\lambda, t)) - \lambda P_t''(x_k(\lambda, t))] \frac{\partial}{\partial t} x_k(\lambda, t) = \frac{\partial}{\partial t} \left[-P_t(x) + \lambda P_t'(x) \right]\Big|_{x=x_k(\lambda,t)}$$

$$= (2t + x_i - x_{i+1}) \left[\prod_{\substack{r=1 \\ r \neq i, i+1}}^{n} (x_k(\lambda, t) - x_r) - \lambda \sum_{\substack{r=1 \\ r \neq i, i+1}}^{n} \prod_{\substack{s=1 \\ s \neq i, i+1, r}}^{n} (x_k(\lambda, t) - x_s) \right]$$

$$= (2t + x_i - x_{i+1}) \left[1 - \lambda \sum_{\substack{r=1 \\ r \neq i, i+1}}^{n} \frac{1}{x_k(\lambda, t) - x_r} \right] \prod_{\substack{r=1 \\ r \neq i, i+1}}^{n} (x_k(\lambda, t) - x_r)$$

$$= \frac{\lambda(2t + x_i - x_{i+1}) P_t(x_k(\lambda, t))}{(x_k(\lambda, t) - x_i - t)(x_k(\lambda, t) - x_{i+1} + t)} \left[\frac{1}{\lambda} - \sum_{\substack{r=1 \\ r \neq i, i+1}}^{n} \frac{1}{x_k(\lambda, t) - x_r} \right]$$

$$= \frac{(2x_k(\lambda, t) - x_i - x_{i+1})(2t + x_i - x_{i+1}) P_t(x_k(\lambda, t))^2}{(x_k(\lambda, t) - x_i - t)^2 (x_k(\lambda, t) - x_{i+1} + t)^2 P_t'(x_k(\lambda, t))}$$

$$= (2x_k(\lambda, t) - x_i - x_{i+1})(2t + x_i - x_{i+1}) F_k(\lambda, t) P_t'(x_k(\lambda, t)).$$

The result follows readily from Lemma 2 since $P_t'(x_k(\lambda, t)) \neq \lambda P_t''(x_k(\lambda, t))$. $\square$

Lemma 4. *Let $m \in \{1, 2, \ldots, n\}$ and $(\lambda, t) \in \mathbb{R} \times (\{0\} \cup I)$. Then*

$$\sum_{k=1}^{m} x_k(\lambda, t) \geq \sum_{k=1}^{m} x_k(\lambda, 0) \text{ if } m \leq i - 1, \ \sum_{k=m}^{n} x_k(\lambda, t) \leq \sum_{k=m}^{n} x_k(\lambda, 0) \text{ if } m \geq i + 2.$$

Proof. If $(\lambda, t) \in \mathbb{R} \times (\{0\} \cup I)$ then (1.1) and Lemma 2 imply that

$$x_k(\lambda, t) < w_k(0, t) < x_{k+1}(0, t) \leq x_i(0, t) < \frac{x_i + x_{i+1}}{2}$$

whenever $k \leq i - 1$ while for $k \geq i + 2$ one gets that

$$x_k(\lambda, t) > w_{k-1}(0, t) > x_{k-1}(0, t) \geq x_{i+1}(0, t) > \frac{x_i + x_{i+1}}{2}.$$

Furthermore, by Lemma 2 one has that $\frac{\partial}{\partial \lambda} x_k(\lambda, t) > 0$ and by (1.2) we know that $F_k(\lambda, t) > 0$ if $k \neq i, i+1$. Therefore, the above inequalities together with Lemma 3 yield

$$\frac{\partial}{\partial t} x_k(\lambda, t) > 0 \text{ if } k \leq i - 1 \text{ and } \frac{\partial}{\partial t} x_k(\lambda, t) < 0 \text{ if } k \geq i + 2.$$

It follows that all the inequalities in the lemma are strict if $(\lambda, t) \in \mathbb{R} \times I$. $\square$

We can now give a proof of Proposition 3:

Proof of Proposition 3. Using the above notations we let $i \in \{1, 2, \ldots, n\}$ and $\sigma \in I$ be such that $Q = \mathcal{T}(i, i+1; \sigma)P$, so that

$$P(\lambda, 0; x) = P(x) - \lambda P'(x) \text{ and } P(\lambda, \sigma; x) = Q(x) - \lambda Q'(x).$$

It is clear that for any $\lambda \in \mathbb{R}$ one has

$$\sum_{k=1}^{n} x_k(\lambda, 0) = \sum_{k=1}^{n} x_k(\lambda, \sigma) = \sum_{k=1}^{n} x_k + n\lambda, \qquad (1.3)$$

where x_k, $1 \le k \le n$, denote as before the zeros of P. By Theorem 4 and (1.3) we see that the relation $Q - \lambda Q' \preccurlyeq P - \lambda P'$ is equivalent to

$$\sum_{k=1}^{m} x_k(\lambda, 0) \le \sum_{k=1}^{m} x_k(\lambda, \sigma), \ 1 \le m \le n-1. \qquad (1.4)$$

These inequalities are trivially true if $\lambda = 0$ and so we may assume that $\lambda \ne 0$. We distinguish two cases:

Case 1: $\lambda > 0$. By Lemma 2 one has $\frac{\partial}{\partial \lambda} x_k(\lambda, t) > 0$. Thus, if $\lambda > 0$ then

$$x_{i+1}(\lambda, t) > x_{i+1}(0, t) = x_{i+1} - t > \frac{x_i + x_{i+1}}{2} \text{ for } t \in [0, \sigma].$$

It follows from Lemma 3 that $\frac{\partial}{\partial t} x_k(\lambda, t) < 0$ if $\lambda > 0$ and $t \in [0, \sigma]$. In particular,

$$x_{i+1}(\lambda, \sigma) < x_{i+1}(\lambda, 0) \text{ if } \lambda > 0. \qquad (1.5)$$

Case 2: $\lambda < 0$. From Lemma 2 again we deduce that in this case one has

$$x_i(\lambda, t) < x_i(0, t) = x_i + t < \frac{x_i + x_{i+1}}{2} \text{ for } t \in [0, \sigma],$$

so that by Lemma 3 we get $\frac{\partial}{\partial t} x_k(\lambda, t) > 0$ if $\lambda < 0$ and $t \in [0, \sigma]$. Hence

$$x_i(\lambda, \sigma) > x_i(\lambda, 0) \text{ if } \lambda < 0. \qquad (1.6)$$

Combining Lemma 4 with (1.5) and (1.6) we see that for any $\lambda \in \mathbb{R} \setminus \{0\}$ one has either

$$\sum_{k=1}^{m} x_k(\lambda, 0) \le \sum_{k=1}^{m} x_k(\lambda, \sigma), \ m \le i, \ \sum_{k=m}^{n} x_k(\lambda, 0) \ge \sum_{k=m}^{n} x_k(\lambda, \sigma), \ m \ge i+2; \text{ or}$$

$$\sum_{k=1}^{m} x_k(\lambda, 0) \le \sum_{k=1}^{m} x_k(\lambda, \sigma), \ m \le i-1, \ \sum_{k=m}^{n} x_k(\lambda, 0) \ge \sum_{k=m}^{n} x_k(\lambda, \sigma), \ m \ge i+1.$$

It is not difficult to see that these relations together with (1.3) yield the inequalities in (1.4), which completes the proof of the proposition. $\qquad \square$

Theorem 2 is now an almost immediate consequence of the above results:

Proof of Theorem 2. In the generic case when both P and Q are strictly hyperbolic polynomials it follows from Proposition 1 that Q may be obtained from P by the successive application of a finite number of simple nondegenerate contractions. Therefore, in this case the theorem follows directly from Proposition 3.

For the general case we let $x_1 \le x_2 \le \ldots \le x_n$ and $y_1 \le y_2 \le \ldots \le y_n$ denote the zeros of P and Q, respectively, counted according to their respective multiplicities. Choose an arbitrary positive number ε and let P_ε and Q_ε be the polynomials with zeros $x_i - (n-i)\varepsilon$, $1 \le i \le n-1$, $x_n + \frac{n(n-1)}{2}\varepsilon$, and $y_i - (n-i)\varepsilon$, $1 \le i \le n-1$, $y_n + \frac{n(n-1)}{2}\varepsilon$, respectively. Note that both P_ε and Q_ε are strictly hyperbolic and that $Q_\varepsilon \preccurlyeq P_\varepsilon$. The above arguments imply that

$$n^{-1}Q'_\varepsilon \preccurlyeq n^{-1}P'_\varepsilon \text{ in } \mathcal{H}_{n-1} \text{ and } Q_\varepsilon + \lambda Q'_\varepsilon \preccurlyeq P_\varepsilon + \lambda P'_\varepsilon, \ \lambda \in \mathbb{R}. \qquad (1.7)$$

Clearly, the zeros and the critical points of P_ε and Q_ε are continuous functions of ε. The desired conclusion follows from Theorem 4 and (1.7) by letting $\varepsilon \to 0$. $\quad\square$

1.2. Applications to differential operators of Laguerre-Pólya type and Appell polynomials.
Theorem 2 has several interesting consequences. In order to state these we need some additional notations and definitions.

Notation 2. Given a nonconstant polynomial $P \in \Pi$ we denote the barycenter of its zeros by $\mathfrak{m}(P)$. Suppose that

$$f(x) = \sum_{k=0}^{\infty} a_k x^k = x^m g(x), \quad x \in \mathbb{C},$$

is an entire function, where m is a nonnegative integer and g is an entire function such that $g(0) \neq 0$. One has a well-defined operator $f(D) \in \operatorname{End} \Pi$ given by

$$f(D)[P](x) = \sum_{k=0}^{\infty} a_k P^{(k)}(x), \quad P \in \Pi,$$

since only finitely many terms in this series are nonzero and so the lack of growth control on the coefficients in the power series expansion of f causes no problems. We associate to f an infinite family of differential operators $\{\mathcal{D}(f,n)\}_{n=m+1}^{\infty}$ defined as follows:

$$\mathcal{D}(f,n) = k_n(f)f(D), \text{ where } k_n(f) = \left[\binom{n}{m}f^{(m)}(0)\right]^{-1}, \, n \geq m+1. \qquad (1.8)$$

Note that these operators are in fact rescalings of $f(D)$ chosen so that if $n \geq m+1$ then $\mathcal{D}(f,n)$ maps monic polynomials of degree n to monic polynomials of degree $n - m$. In particular, if $m = 0$ then all operators $\mathcal{D}(f,n)$, $n \in \mathbb{N}$, coincide with $f(0)^{-1}f(D)$ and preserve the class of monic polynomials of degree d for any $d \geq 0$.

Definition 3. A real entire function φ is said to be in the *Laguerre-Pólya class*, $\varphi \in \mathcal{LP}$, if it can be expressed in the form

$$\varphi(x) = cx^m e^{-a^2 x^2 + bx} \prod_{k=1}^{\infty}(1 - \alpha_k x)e^{\alpha_k x}, \quad x \in \mathbb{C}, \qquad (1.9)$$

where $a, b, c, \alpha_k \in \mathbb{R}$, $c \neq 0$, m is a nonnegative integer, $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$ and where, by the usual convention, the canonical product reduces to 1 if $\alpha_k = 0$ for all $k \in \mathbb{N}$. An operator $T \in \operatorname{End} \Pi$ is said to be a *differential operator of Laguerre-Pólya type* if $T = \varphi(D)$, where $\varphi \in \mathcal{LP}$.

Notation 3. Let $\mathcal{LP}_0 := \{\varphi \in \mathcal{LP} \mid \varphi(0) \neq 0\}$. For $m \in \mathbb{N}$ we set

$$\mathcal{LP}_m = x^m \mathcal{LP}_0 = \left\{\varphi \in \mathcal{LP} \mid \varphi^{(k)}(0) = 0, \, 0 \leq k \leq m-1, \, \varphi^{(m)}(0) \neq 0\right\}.$$

Clearly, $\mathcal{LP}$ is a commutative monoid under ordinary multiplication of functions. Actually, $\mathcal{LP}$ may be viewed as a $\mathbb{Z}_+$-graded monoid, where $\mathbb{Z}_+$ denotes the additive monoid of nonnegative integers. Indeed, note that $\mathcal{LP}_0$ is a submonoid of $\mathcal{LP}$ which acts on $\mathcal{LP}_m$ for each $m \in \mathbb{Z}_+$ and that $\mathcal{LP}$ decomposes into a disjoint union

$$\mathcal{LP} = \bigcup_{m=0}^{\infty} \mathcal{LP}_m \text{ with } \mathcal{LP}_{m_1} \cdot \mathcal{LP}_{m_2} = \mathcal{LP}_{m_1+m_2} \text{ for } m_1, m_2 \in \mathbb{Z}_+. \qquad (1.10)$$

As we already pointed out in the introduction, by a classical theorem of Pólya one knows that all differential operators of Laguerre-Pólya type map hyperbolic polynomials to hyperbolic polynomials. By using Theorem 2 one can actually show that all such operators are in fact natural preservers of the spectral order:

Corollary 1. *Let $m, n \in \mathbb{Z}_+$ with $n \geq m+1$ and $\varphi \in \mathcal{LP}_m$. If $P, Q \in \mathcal{H}_n$ are such that $Q \preccurlyeq P$ then $\mathcal{D}(\varphi, n)[Q] \preccurlyeq \mathcal{D}(\varphi, n)[P]$ in $\mathcal{H}_{n-m}$.*

Remark 3. It is clear that if $\varphi \in \mathcal{LP}_m$ then $\mathcal{D}(\varphi, m)[P] \equiv 1$ for all $P \in \mathcal{H}_m$ while $\mathcal{D}(\varphi, n)[P] \equiv 0$ if $P \in \mathcal{H}_n$ with $n \leq m - 1$. This is the reason why we impose the condition $n \geq m + 1$ both in Corollary 1 and Corollary 5 of §2.

To prove Corollary 1 we need to establish first the following result.

Lemma 5. *Let $n \geq 2$ and $P, Q \in \mathcal{H}_n$ with $Q \preccurlyeq P$. Then $n^{-1}Q' \preccurlyeq n^{-1}P'$ in $\mathcal{H}_{n-1}$.*

Proof. It is enough to prove the lemma in the generic case when P and Q are strictly hyperbolic polynomials and Q is a simple nondegenerate contraction of P (the general case follows from this one by arguing as in the proof of Theorem 2). Let then $Q = \mathcal{T}(i, i+1; \sigma)P$, where $\sigma \in I$ and $i \in \{1, 2, \ldots, n\}$. Using Notation 1 we may write

$$P(\lambda, 0; x) = P(x) - \lambda P'(x) = \prod_{k=1}^{n}(x - x_k(\lambda, 0)), \;\; P'(\lambda, 0; x) = n\prod_{l=1}^{n-1}(x - w_l(\lambda, 0)),$$

$$P(\lambda, \sigma; x) = Q(x) - \lambda Q'(x) = \prod_{k=1}^{n}(x - x_k(\lambda, \sigma)), \;\; P'(\lambda, \sigma; x) = n\prod_{l=1}^{n-1}(x - w_l(\lambda, \sigma)).$$

By Proposition 3 we know that $P(\lambda, \sigma; x) \preccurlyeq P(\lambda, 0; x)$, so that (1.4) is valid. Therefore, if we let $\lambda \to \infty$ in (1.4) and use the second part of Lemma 2 we obtain

$$\sum_{j=1}^{m} w_j(0,0) \leq \sum_{j=1}^{m} w_j(0,\sigma), \; 1 \leq m \leq n - 1. \tag{1.11}$$

Since Q is a contraction of P one has $Q \preccurlyeq P$, so that $\mathfrak{m}(Q) = \mathfrak{m}(P)$ and thus $\mathfrak{m}(Q') = \mathfrak{m}(P')$. This shows that the inequality in (1.11) corresponding to $m = n-1$ is actually an equality, which by Theorem 4 proves the lemma. $\qquad\square$

Proof of Corollary 1. Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two unordered n-tuples of real numbers and set

$$d(X, Y) = \min_{\pi \in \Sigma_n} \max_{1 \leq i \leq n} \left| x_i - y_{\pi(i)} \right|.$$

This is the so-called *optimal matching distance* between the unordered n-tuples X and Y. It is not difficult to see that d defines a metric on the quotient space $\mathbb{R}^n/\Sigma_n$ of all such n-tuples and therefore also on the manifold $\mathcal{H}_n$ in view of (0.1).

We use the rearrangement-free characterization of the spectral order given in Theorem 1 (i) in the following way: to any function $f : \mathbb{R} \to \mathbb{R}$ we associate a function $\tilde{f} : \mathbb{R}^n/\Sigma_n \to \mathbb{R}$ by setting

$$\tilde{f}(X) = \sum_{i=1}^{n} f(x_i) \text{ for } X = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n/\Sigma_n. \tag{1.12}$$

If f is convex then Theorem 1 (i) asserts that $\tilde{f}(X) \leq \tilde{f}(Y)$ whenever $X \prec Y$, that is, $\tilde{f}$ is a *Schur-convex* function (cf. [MO, Ch. 3]). Thus $X \prec Y$ if and only if $\tilde{f}(X) \leq \tilde{f}(Y)$ for any function $\tilde{f}$ as in (1.12) associated to a convex function f.

Assume now that $P, Q \in \mathcal{H}_n$ are such that $Q \preccurlyeq P$ and let $\varphi \in \mathcal{LP}_m$, where $m \in \mathbb{Z}_+$, $m \leq n - 1$. Suppose that φ is as in (1.9) with Maclaurin expansion

$$\varphi(x) = \sum_{k=m}^{\infty} a_k x^k, \quad x \in \mathbb{C}.$$

For $j \in \mathbb{N}$ let $\tau_j := b + \sum_{\nu=1}^{j} \alpha_\nu$ and define the following polynomials:

$$\varphi_j(x) = cx^m \left(1 - \frac{ax}{\sqrt{j}}\right)^j \left(1 + \frac{ax}{\sqrt{j}}\right)^j \left(1 + \frac{\tau_j x}{n_j}\right)^{n_j} \prod_{\nu=1}^{j} (1 - \alpha_\nu x). \qquad (1.13)$$

It is a well-known fact that if one chooses $\{n_j\}_{j \in \mathbb{N}}$ as a sequence of integers growing sufficiently fast to infinity as $j \to \infty$ then the sequence of hyperbolic polynomials $\{\varphi_j\}_{j \in \mathbb{N}}$ satisfies $\varphi_j \rightrightarrows \varphi$ as $j \to \infty$, where $\rightrightarrows$ denotes uniform convergence on all compact subsets of $\mathbb{C}$ (see, e.g., [L, Ch. 8]). Therefore, if we let $N_j := \deg \varphi_j$ and write the polynomial φ_j as

$$\varphi_j(x) = \sum_{k=m}^{\infty} a_{j,k} x^k, \quad x \in \mathbb{C},$$

with $a_{j,k} = 0$ for $k \geq N_j + 1$ then it follows from Cauchy's integral formula that $\lim_{j \to \infty} a_{j,k} = a_k$ for all $k \geq m$. This implies that for any fixed polynomial $R \in \Pi$ with $\deg R = n$ one has

$$\varphi_j(D)[R] = \sum_{k=m}^{n} a_{j,k} R^{(k)} \rightrightarrows \sum_{k=m}^{n} a_k R^{(k)} = \varphi(D)[R] \text{ as } j \to \infty.$$

In particular, $\mathcal{D}(\varphi_j, n)[P] \rightrightarrows \mathcal{D}(\varphi, n)[P]$ and $\mathcal{D}(\varphi_j, n)[Q] \rightrightarrows \mathcal{D}(\varphi, n)[Q]$ as $j \to \infty$, so that

$$\begin{aligned} d\big(\mathcal{Z}(\mathcal{D}(\varphi_j, n)[P]), \mathcal{Z}(\mathcal{D}(\varphi, n)[P])\big) &\longrightarrow 0 \text{ and} \\ d\big(\mathcal{Z}(\mathcal{D}(\varphi_j, n)[Q]), \mathcal{Z}(\mathcal{D}(\varphi, n)[Q])\big) &\longrightarrow 0 \text{ as } j \longrightarrow \infty. \end{aligned} \qquad (1.14)$$

On the other hand, by Theorem 2 and Lemma 5 we know that

$$\mathcal{Z}(\mathcal{D}(\varphi_j, n)[Q]) \prec \mathcal{Z}(\mathcal{D}(\varphi_j, n)[P]) \text{ in } \mathbb{R}^{n-m}/\Sigma_{n-m} \text{ for } j \in \mathbb{N}.$$

Thus, if f is a real-valued convex function on $\mathbb{R}$ and $\tilde{f}$ is as in (1.12) then

$$\tilde{f}\big(\mathcal{Z}(\mathcal{D}(\varphi_j, n)[Q])\big) \leq \tilde{f}\big(\mathcal{Z}(\mathcal{D}(\varphi_j, n)[P])\big) \text{ for } j \in \mathbb{N}. \qquad (1.15)$$

Since f is convex on $\mathbb{R}$ it is also continuous there and so $\tilde{f}$ is a continuous function on $\mathbb{R}^n/\Sigma_n$. Therefore, by letting $j \to \infty$ in (1.14) and (1.15) we obtain

$$\tilde{f}\big(\mathcal{Z}(\mathcal{D}(\varphi, n)[Q])\big) \leq \tilde{f}\big(\mathcal{Z}(\mathcal{D}(\varphi, n)[P])\big).$$

As explained above, this implies that

$$\mathcal{Z}(\mathcal{D}(\varphi, n)[Q]) \prec \mathcal{Z}(\mathcal{D}(\varphi, n)[P]) \text{ in } \mathbb{R}^{n-m}/\Sigma_{n-m}.$$

Hence $\mathcal{D}(\varphi, n)[Q] \preccurlyeq \mathcal{D}(\varphi, n)[P]$ in $\mathcal{H}_{n-m}$, which completes the proof. $\qquad \square$

Notation 4. Define the following monoids of linear operators:

$$\mathcal{A} = \bigcap_{n=0}^{\infty} \mathcal{A}_n, \text{ where } \mathcal{A}_n = \big\{T \in \text{End}\,\Pi \mid T(\mathcal{H}_n) \subseteq \mathcal{H}_n\big\}, \, n \in \mathbb{Z}_+. \qquad (1.16)$$

Note that $\mathcal{A}_n$ is the largest submonoid of $\text{End}\,\Pi$ consisting of linear operators that act on $\mathcal{H}_n$ for fixed $n \in \mathbb{Z}_+$, while $\mathcal{A}$ is the largest submonoid of $\text{End}\,\Pi$ acting on each of the manifolds $\mathcal{H}_n$, $n \in \mathbb{Z}_+$.

In [CPP, Theorem 1] it was shown that

$$\mathcal{A} = \{\varphi(D) \mid \varphi \in \mathcal{LP}, \, \varphi(0) = 1\} \subset \mathcal{LP}_0. \qquad (1.17)$$

From Corollary 1 and (1.17) we deduce that all operators in $\mathcal{A}$ are *isotonic* (see Definition 4 below) with respect to the spectral order on $\mathcal{H}_n$ for any $n \in \mathbb{N}$:

Corollary 2. *If $n \geq 1$ and $P, Q \in \mathcal{H}_n$ are such that $Q \preccurlyeq P$ then $T[Q] \preccurlyeq T[P]$ for all operators $T \in \mathcal{A}$.* $\qquad \square$

Yet another consequence of Corollary 1 is that the sequence of nonconstant Appell polynomials associated to any given function in the Laguerre-Pólya class may be characterized by means of a global minimum property with respect to the spectral order. Indeed, let $n \in \mathbb{N}$ and consider the following submanifold of $\mathcal{H}_n$:

$$\mathcal{H}_n^0 = \{P \in \mathcal{H}_n \mid \mathfrak{m}(P) = 0\}. \tag{1.18}$$

Given $\varphi \in \mathcal{LP}$ and $n \in \mathbb{Z}_+$ one defines the n-th *Appell polynomial* g_n^* associated with φ by $g_n^*(x) = \varphi(D)[x^n]$ (see, e.g., [CC1]). Recall the decomposition of $\mathcal{LP}$ from (1.10) and assume that $\varphi \in \mathcal{LP}_m$ for some $m \in \mathbb{Z}_+$. Clearly, g_n^* is a nonconstant polynomial if and only if $n \geq m + 1$ (cf. Remark 3). Corollary 1, Theorem 4, and the fact that $x^n \preccurlyeq P(x)$ for any $P \in \mathcal{H}_n^0$, $n \in \mathbb{N}$, yield the following:

Corollary 3. *Let $m \in \mathbb{Z}_+$ and $\varphi \in \mathcal{LP}_m$. If $n \geq m + 1$ then the monic polynomial $k_n(\varphi)g_n^*$ is the (unique) global minimum of the poset $\big(\mathcal{D}(\varphi, n)[\mathcal{H}_n^0], \preccurlyeq\big)$, where $\mathcal{D}(\varphi, n)[\mathcal{H}_n^0] := \{\mathcal{D}(\varphi, n)[P] \mid P \in \mathcal{H}_n^0\}$, $k_n(\varphi)$ is as in (1.8) and g_n^* is the n-th Appell polynomial associated with φ.* $\square$

In view of Theorems 1 and 4, Corollary 3 admits the following geometrical interpretation: up to a factor $k_n(\varphi)$ the n-th Appell polynomial associated with φ coincides with the (unique) polynomial in the image set $\mathcal{D}(\varphi, n)[\mathcal{H}_n^0]$ whose zeros are less spread out than the zeros of any other polynomial in this set.

Remark 4. A systematic investigation of the topological properties of $\mathcal{H}_n$ and $\mathcal{H}_n^0$ was initiated by Arnold in [Ar]. These manifolds have since been extensively studied in singularity theory and related topics.

2. THEOREM 3 AND SOME CONSEQUENCES

2.1. **Proof of Theorem 3.** The result holds trivially for $n = 1$ and so we may assume that $n \geq 2$. As in §1, we start with a strictly hyperbolic polynomial $P \in \mathcal{H}_n$ given by

$$P(x) = \prod_{i=1}^{n}(x - x_i) \text{ and } P'(x) = n\prod_{j=1}^{n-1}(x - w_j)$$

with $x_1 < w_1 < x_2 < \ldots < x_{n-1} < w_{n-1} < x_n$ and we define the following pencils of polynomials:

$$P_\lambda(x) = P(x) - \lambda P'(x) \text{ and } P_\lambda'(x) = P'(x) - \lambda P''(x), \quad \lambda \in \mathbb{R}.$$

Denote the zeros of P_λ and P_λ' by $x_i(\lambda)$, $1 \leq i \leq n$, and $w_j(\lambda)$, $1 \leq j \leq n - 1$, respectively. If we assume that these are labeled so that $x_i(0) = x_i$, $1 \leq i \leq n$, and $w_j(\lambda) = w_j$, $1 \leq j \leq n - 1$, then by letting $t = 0$ in (1.1) we see that

$$x_1(\lambda) < w_1(\lambda) < x_2(\lambda) < \ldots < x_{n-1}(\lambda) < w_{n-1}(\lambda) < x_n(\lambda) \tag{2.1}$$

for all $\lambda \in \mathbb{R}$. The following proposition is the key step in the proof of Theorem 3.

Proposition 4. *If P is as above then each of the functions $f_m : \mathbb{R} \to \mathbb{R}$ given by*

$$f_m(\lambda) = \sum_{i=1}^{m}(x_i(\lambda) - \lambda), \quad 1 \leq m \leq n - 1,$$

is increasing on $(-\infty, 0]$ and decreasing on $[0, \infty)$.

The proof of Proposition 4 is based on two lemmas:

Lemma 6. *Let $1 \leq j \leq n - 1$ and $\lambda \in \mathbb{R}$. Then*

$$\sum_{i=1}^{m} \frac{1}{x_i(\lambda) - w_j(\lambda)} < 0$$

for all $m \in \{1, \ldots, n - 1\}$.

Proof. If $m \leq j$ then for each $i \leq m$ one has $x_i(\lambda) \leq x_m(\lambda) < w_j(\lambda)$ by (2.1), so that in this case all terms in the sum are negative. Assume that $m \geq j + 1$. Then

$$0 = \frac{P'_\lambda(w_j(\lambda))}{P_\lambda(w_j(\lambda))} = \sum_{i=1}^{n} \frac{1}{w_j(\lambda) - x_i(\lambda)} = \sum_{i=1}^{m} \frac{1}{w_j(\lambda) - x_i(\lambda)} + \sum_{i=m+1}^{n} \frac{1}{w_j(\lambda) - x_i(\lambda)}.$$

Thus

$$\sum_{i=1}^{m} \frac{1}{x_i(\lambda) - w_j(\lambda)} = \sum_{i=m+1}^{n} \frac{1}{w_j(\lambda) - x_i(\lambda)} < 0$$

since (2.1) implies that $x_i(\lambda) \geq x_{m+1}(\lambda) > w_j(\lambda)$ if $i \geq m + 1$. $\qquad\square$

Lemma 7. *If $1 \leq j \leq n - 1$ and $\lambda \in \mathbb{R}$ then*

$$w'_j(\lambda) = \frac{P''(w_j(\lambda))}{P''_\lambda(w_j(\lambda))} > 0,$$

where $P''_\lambda(x) = \frac{\partial}{\partial x} P'_\lambda(x)$.

Proof. Apply Lemma 2 to $P'(\lambda, t, w_j(\lambda, t))$, $1 \leq j \leq n - 1$, and set $t = 0$. $\qquad\square$

Proof of Proposition 4. Using Lemma 2 and a partial fractional decomposition we get

$$x'_i(\lambda) - 1 = \frac{\lambda P''(x_i(\lambda))}{P'_\lambda(x_i(\lambda))} = \sum_{j=1}^{n-1} \frac{P''(w_j(\lambda))}{P''_\lambda(w_j(\lambda))} \frac{\lambda}{x_i(\lambda) - w_j(\lambda)} = \sum_{j=1}^{n-1} \frac{\lambda w'_j(\lambda)}{x_i(\lambda) - w_j(\lambda)}.$$

Therefore, if $1 \leq m \leq n - 1$ then

$$f'_m(\lambda) = \sum_{i=1}^{m} (x'_i(\lambda) - 1) = \lambda \sum_{j=1}^{n-1} \sum_{i=1}^{m} \frac{w'_j(\lambda)}{x_i(\lambda) - w_j(\lambda)}. \tag{2.2}$$

Lemmas 6 and 7 imply that

$$\sum_{i=1}^{m} \frac{w'_j(\lambda)}{x_i(\lambda) - w_j(\lambda)} < 0, \quad \lambda \in \mathbb{R},$$

which together with (2.2) shows that $\lambda f'_m(\lambda) < 0$ if $\lambda \neq 0$, as required. $\qquad\square$

Theorem 3 is now a straightforward consequence of Theorem 4 and the following result.

Proposition 5. *Let $P \in \mathcal{H}_n$ and set $P_\lambda(x) = P(x) - \lambda P'(x)$, where $\lambda \in \mathbb{R}$. For any fixed λ denote the zeros of P_λ by $x_i(\lambda)$, $1 \leq i \leq n$, and arrange these so that $x_1(\lambda) \leq \ldots \leq x_n(\lambda)$. Given $m \in \{1, 2, \ldots, n\}$ we define a function $f_m : \mathbb{R} \to \mathbb{R}$ by*

$$f_m(\lambda) = \sum_{i=1}^{m} (x_i(\lambda) - \lambda).$$

If $1 \leq m \leq n - 1$ then f_m is nondecreasing on $(-\infty, 0]$ and it is nonincreasing on $[0, \infty)$. Moreover, f_n is a constant function on $\mathbb{R}$.

Proof. The first assertion follows from Proposition 4 since P may be approximated by strictly hyperbolic polynomials in $\mathcal{H}_n$ uniformly on compact subsets of $\mathbb{C}$. Indeed, if $\varepsilon \in \mathbb{R} \setminus \{0\}$ then $\hat{P}_\varepsilon(x) := (1 - \varepsilon D)^{n-1} P(x)$ is a strictly hyperbolic polynomial in $\mathcal{H}_n$ (cf., e.g., [CC2, Lemma 4.2]). It is clear that $\hat{P}_\varepsilon \rightrightarrows P$ as $\varepsilon \to 0$. The second statement follows from the fact that $f_n(\lambda) = \sum_{i=1}^{n} x_i$ for all $\lambda \in \mathbb{R}$, where x_i, $1 \leq i \leq n$, are the zeros of P. $\qquad\square$

Remark 5. Proposition 5 has recently been extended to arbitrary hyperbolic polynomial pencils in [BP], where it was furthermore shown that f_m, $1 \leq m \leq n-1$, are actually concave functions on $\mathbb{R}$. Note that by [B, Theorem 4] these partial sums cannot have a common local maximum unless the polynomial pencil under consideration is of logarithmic derivative type, i.e., of the form $P - \lambda P'$, $\lambda \in \mathbb{R}$.

Corollary 4. *Let $\lambda_1, \lambda_2 \in \mathbb{R}$ be such that $\lambda_1 \lambda_2 \geq 0$ and $|\lambda_1| \leq |\lambda_2|$. If $m, n \in \mathbb{Z}_+$ with $n \geq \max(2, m+1)$ then for any $P \in \mathcal{H}_n$ one has*

$$\binom{n}{m}^{-1} D^m(1 - \lambda_1 D)e^{\lambda_1 D}P \preccurlyeq \binom{n}{m}^{-1} D^m(1 - \lambda_2 D)e^{\lambda_2 D}P \ \ in \ \mathcal{H}_{n-m}.$$

In particular, if $s_1, s_2 \in \mathbb{R}$ satisfy $s_1 s_2 \geq 0$ and $|s_1| \leq |s_2|$ then

$$\binom{n}{m}^{-1} D^m(1 - s_1 \lambda D)e^{s_1 \lambda D}P \preccurlyeq \binom{n}{m}^{-1} D^m(1 - s_2 \lambda D)e^{s_2 \lambda D}P$$

$$e^{-s_1^2 \lambda^2 D^2}P \preccurlyeq e^{-s_2^2 \lambda^2 D^2}P$$

for all $P \in \mathcal{H}_n$ and $\lambda \in \mathbb{R}$.

Proof. The first relation is an immediate consequence of Theorem 4, Proposition 5 and repeated use of Lemma 5 since $(1 - \lambda D)e^{\lambda D}P(x) = P(x + \lambda) - \lambda P'(x + \lambda)$ for all $\lambda \in \mathbb{R}$. Setting $\lambda_i = s_i \lambda$, $i = 1, 2$, one gets the second relation. Let $j \in \mathbb{N}$ and define a function

$$\psi_j(x) = \left(1 - \frac{\lambda^2 x^2}{j}\right)^j = \left[\left(1 - \frac{\lambda x}{\sqrt{j}}\right)e^{\frac{\lambda x}{\sqrt{j}}}\right]^j \left[\left(1 + \frac{\lambda x}{\sqrt{j}}\right)e^{-\frac{\lambda x}{\sqrt{j}}}\right]^j,$$

where λ is a fixed real number. Clearly, the second relation implies that for any $P \in \mathcal{H}_n$ and $j \in \mathbb{N}$ one has $\psi_j(s_1 D)[P] \preccurlyeq \psi_j(s_2 D)[P]$. Moreover, from $\psi_j(x) \rightrightarrows e^{-\lambda^2 x^2}$ as $j \to \infty$ one easily gets $\psi_j(s_i D)[P] \rightrightarrows e^{-s_i^2 \lambda^2 D^2}P$ for $i = 1, 2$. The third relation is obtained by letting $j \to \infty$. $\qquad\square$

2.2. **Orbits of hyperbolic polynomials.** Theorem 3 and Corollary 4 allow us to study the orbits of hyperbolic polynomials under the action of differential operators of Laguerre-Pólya type. To do this we need some new notation.

Notation 5. Let l^∞ denote the Banach algebra of bounded real sequences of the form $\{s_i\}_{i=0}^\infty$. We endow l^∞ with a partial ordering $\leqslant$ defined as follows: given two elements $\mathbf{s} = \{s_i\}_{i=0}^\infty$ and $\mathbf{t} = \{t_i\}_{i=0}^\infty$ of l^∞ we set $\mathbf{s} \leqslant \mathbf{t}$ if $|s_i| \leq |t_i|$ and $s_i t_i \geq 0$ for all $i \in \mathbb{Z}_+$. For fixed $\mathbf{s} = \{s_i\}_{i=0}^\infty \in l^\infty$, $m \in \mathbb{Z}_+$ and a function $\varphi \in \mathcal{LP}_m$ of the form (1.9) we define the $\mathbf{s}$-*deformation* of φ to be

$$\varphi^{\mathbf{s}}(x) = cx^m e^{-s_0^2 a^2 x^2 + bx} \prod_{k=1}^\infty (1 - s_k \alpha_k x)e^{s_k \alpha_k x}, \quad x \in \mathbb{C}. \tag{2.3}$$

Note that $\varphi^{\mathbf{s}} \in \mathcal{LP}_m$ and so (2.3) defines an action of l^∞ on $\mathcal{LP}_m$ for any $m \in \mathbb{Z}_+$

$$l^\infty \times \mathcal{LP}_m \longrightarrow \mathcal{LP}_m$$
$$(\mathbf{s}, \varphi) \longmapsto \mathbf{s} \cdot \varphi := \varphi^{\mathbf{s}} \tag{2.4}$$

by means of which we associate to any $\varphi \in \mathcal{LP}_m$ an infinite-parameter family of deformations of the operator $\varphi(D)$, namely

$$\mathcal{F}_\varphi := \{\mathcal{D}(\varphi^{\mathbf{s}}, n) \mid \mathbf{s} \in l^\infty, n \in \mathbb{N}, n \geq m+1\},$$

where $\mathcal{D}(\varphi^{\mathbf{s}}, n)$ is as in (1.8).

The operator families $\mathcal{F}_\varphi$ satisfy the following global monotony property with respect to the partial orderings $\leqslant$ on l^∞ and $\preccurlyeq$ on $\mathcal{H}_n$, $n \in \mathbb{Z}_+$, respectively:

Corollary 5. *Let $m, n \in \mathbb{Z}_+$ with $n \geq m + 1$ and $\varphi \in \mathcal{LP}_m$. If $\mathbf{s}, \mathbf{t} \in l^\infty$ are such that $\mathbf{s} \leqslant \mathbf{t}$ then $\mathcal{D}\left(\varphi^{\mathbf{s}}, n\right)[P] \preccurlyeq \mathcal{D}\left(\varphi^{\mathbf{t}}, n\right)[P]$ in $\mathcal{H}_{n-m}$ for any $P \in \mathcal{H}_n$.*

Proof. Let us fix $\mathbf{s} = \{s_i\}_{i=0}^\infty \in l^\infty$ and $\mathbf{t} = \{t_i\}_{i=0}^\infty \in l^\infty$ such that $\mathbf{s} \leqslant \mathbf{t}$. Given $m, n \in \mathbb{Z}_+$ with $n \geq \max(2, m+1)$ and $\varphi \in \mathcal{LP}_m$ as in (1.9) we approximate $\varphi^{\mathbf{s}}(x)$ and $\varphi^{\mathbf{t}}(x)$ uniformly on compact subsets of $\mathbb{C}$ by means of the functions

$$\varphi_j^{\mathbf{s}}(x) = cx^m e^{-s_0^2 a^2 x^2 + bx} \prod_{k=1}^{j}(1 - s_k \alpha_k x) e^{s_k \alpha_k x} \text{ and}$$

$$\varphi_j^{\mathbf{t}}(x) = cx^m e^{-t_0^2 a^2 x^2 + bx} \prod_{k=1}^{j}(1 - t_k \alpha_k x) e^{t_k \alpha_k x},$$

respectively, where $j \in \mathbb{N}$. By Corollary 4 we know that

$$\mathcal{D}(\varphi_j^{\mathbf{s}}, n)[P] \preccurlyeq \mathcal{D}(\varphi_j^{\mathbf{t}}, n)[P] \text{ in } \mathcal{H}_{n-m} \tag{2.5}$$

for arbitrarily fixed $P \in \mathcal{H}_n$ and $j \in \mathbb{N}$. Standard arguments involving the uniform convergence of the above sequences of functions similar to those given in the proof of Corollary 1 show that $\mathcal{D}(\varphi_j^{\mathbf{s}}, n)[P] \rightrightarrows \mathcal{D}(\varphi^{\mathbf{s}}, n)[P]$ and $\mathcal{D}(\varphi_j^{\mathbf{t}}, n)[P] \rightrightarrows \mathcal{D}(\varphi^{\mathbf{t}}, n)[P]$ as $j \to \infty$. The desired result follows from (2.5) by letting $j \to \infty$. $\quad\square$

Recall from (1.17) that $\mathcal{A}$ is the largest submonoid of $\operatorname{End}\Pi$ acting on each of the manifolds $\mathcal{H}_n$, $n \in \mathbb{Z}_+$. We define a binary relation on $\mathcal{A}$ which by abuse of notation we denote again by $\preccurlyeq$ in the following manner: given $T_1, T_2 \in \mathcal{A}$ set $T_1 \preccurlyeq T_2$ if $T_1[P] \preccurlyeq T_2[P]$ for all $P \in \mathcal{H}_n$, $n \in \mathbb{N}$.

Lemma 8. *The pair $(\mathcal{A}, \preccurlyeq)$ is a poset.*

Proof. Clearly, $\preccurlyeq$ inherits the reflexivity and transitivity properties from the partial orderings on the posets $(\mathcal{H}_n, \preccurlyeq)$, $n \in \mathbb{Z}_+$. Assume that $T_1, T_2 \in \mathcal{A}$ are such that $T_1 \preccurlyeq T_2$ and $T_2 \preccurlyeq T_1$. By (1.16) we may write $T_i = \varphi_i(D)$, where $\varphi_i \in \mathcal{LP}$ with $\varphi_i(0) = 1$, $i = 1, 2$. In particular, $\varphi_1(D)[x^n] \preccurlyeq \varphi_2(D)[x^n]$ and $\varphi_2(D)[x^n] \preccurlyeq \varphi_1(D)[x^n]$, $n \in \mathbb{Z}_+$. Since $(\mathcal{H}_n, \preccurlyeq)$ is a poset for all $n \in \mathbb{Z}_+$ we deduce that the sequences of Appell polynomials associated to φ_1 and φ_2 must coincide. It follows that $\varphi_1 = \varphi_2$ and thus $T_1 = T_2$, which shows that $\preccurlyeq$ is also antisymmetric. $\quad\square$

From Corollary 5 we deduce the following compatibility relation between the posets $(l^\infty, \leqslant)$ and $(\mathcal{A}, \preccurlyeq)$.

Corollary 6. *If $T \in \mathcal{A}$ and $\mathbf{s}, \mathbf{t} \in l^\infty$ with $\mathbf{s} \leqslant \mathbf{t}$ then $\mathbf{s} \cdot T \preccurlyeq \mathbf{t} \cdot T$.* $\quad\square$

Let $\mathcal{LP}'$ be the class of entire functions of the form

$$\varphi(x) = cx^m e^{-a^2 x^2} \prod_{k=1}^{\infty}(1 - \alpha_k x) e^{\alpha_k x}, \quad x \in \mathbb{C}, \tag{2.6}$$

where $a, c, \alpha_k \in \mathbb{R}$, $c \neq 0$, $m \in \mathbb{Z}_+$ and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$, so that $\mathcal{LP}' \subset \mathcal{LP}$. For $m \in \mathbb{Z}_+$ we set $\mathcal{LP}'_m = \mathcal{LP}' \cap \mathcal{LP}_m$. By taking constant sequences $\mathbf{s} = \{s\}_{i=0}^\infty$ and $\mathbf{t} = \{t\}_{i=0}^\infty$ in Corollary 5 we obtain the following generalization of Theorems 1.4 and 1.6 in [BS].

Corollary 7. *Let $n \in \mathbb{N}$ and $\varphi \in \mathcal{LP}'$ with $\varphi(0) = 1$. If $s, t \in \mathbb{R}$ are such that $|s| \leq |t|$ and $st \geq 0$ then $\varphi(sD)[P] \preccurlyeq \varphi(tD)[P]$ for any $P \in \mathcal{H}_n$.* $\quad\square$

Let $\mathcal{A}'$ be the submonoid of $\mathcal{A}$ consisting of all operators that preserve the barycenter of the zeros of any nonconstant polynomial. Then by (1.17) one has

$$\mathcal{A}' = \left\{T \in \mathcal{A} \mid \mathfrak{m}\left(T(P)\right) = \mathfrak{m}(P) \text{ if } P \in \Pi, \deg P \geq 1\right\}$$
$$= \left\{\varphi(D) \mid \varphi \in \mathcal{LP}', \varphi(0) = 1\right\} \subset \mathcal{LP}'_0.$$

Setting $s = 0$ and $t = 1$ in Corollary 7 we deduce that any nonconstant monic hyperbolic polynomial is the global minimum of its $\mathcal{A}'$-orbit. In this way we recover Theorem 6 of [B]:

Corollary 8. *If $n \in \mathbb{N}$ then $P \preccurlyeq T[P]$ for all $P \in \mathcal{H}_n$ and $T \in \mathcal{A}'$.* $\hfill\square$

Finally, let us note that some of the properties established above may be restated by using the following terminology of set-theoretic topology:

Definition 4. An operator T on a poset $(\mathcal{X}, \leq)$ is called *isotonic* if $T[x] \leq T[y]$ whenever $x, y \in \mathcal{X}$ are such that $x \leq y$ while T is said to be *extensive* (or *expanding*) if $x \leq T[x]$ for any $x \in \mathcal{X}$. An operator on $(\mathcal{X}, \leq)$ which is idempotent, isotonic and extensive with respect to $\leq$ is called a *closure operator* on $\mathcal{X}$.

For instance, Corollary 1 asserts that essentially all differential operators of Laguerre-Pólya type are isotonic on each of the posets $(\mathcal{H}_n, \preccurlyeq)$, $n \in \mathbb{N}$, while Corollary 7 shows that the monoid $\mathcal{A}'$ consists of differential operators of Laguerre-Pólya type which are extensive with respect to the spectral order.

Remark 6. The proofs of Theorems 2 and 3 were essentially based on a detailed analysis of the dynamics of the zeros and critical points of strictly hyperbolic polynomials under the action of differential operators of Laguerre-Pólya type. There are many known examples of such operators that actually map any hyperbolic polynomial to a *strictly* hyperbolic polynomial (cf., e.g., [CC1, CC2]). For instance, if Q is a hyperbolic polynomial of degree n and $b \in \mathbb{R}$ then $e^{bD}Q(D)[P]$ is strictly hyperbolic whenever P is a hyperbolic polynomial of degree at most $n + 1$. Moreover, if $\varphi(x)$ is a transcendental function in the Laguerre-Pólya class which is not of the form $Q(x)e^{bx}$ for some hyperbolic polynomial Q and $b \in \mathbb{R}$ then a theorem of Pólya asserts that $\varphi(D)[P]$ is strictly hyperbolic for any hyperbolic polynomial P. In particular, this holds if $\varphi(x) = e^{-a^2 x}$ with $a \in \mathbb{R} \setminus \{0\}$.

3. Further results and related topics

In this section we state several other consequences of Theorems 2 and 3 and discuss some related problems.

3.1. **The distribution of zeros of hyperbolic polynomials.** The results given in §1–2 have interesting applications to the distribution and the relative geometry of zeros of hyperbolic polynomials and their images under the action of differential operators of Laguerre-Pólya type. Recall from §1 that a function $\Phi : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex if $\Phi(X) \leq \Phi(Y)$ whenever $X, Y \in \mathbb{R}^n$ are such that $X \prec Y$. Given a polynomial $P \in \Pi$ of degree $n \geq 1$ we denote its zeros by $x_i(P)$, $1 \leq i \leq n$. Then Theorems 1 and Corollary 1 yield the following result.

Corollary 9. *Let $n, m \in \mathbb{Z}_+$ with $n \geq m + 1$. If $\varphi \in \mathcal{LP}_m$ and $\Phi : \mathbb{R}^{n-m} \to \mathbb{R}$ is a Schur-convex function then*

$$\Phi\big(x_1(\varphi(D)[Q]), \ldots, x_{n-m}(\varphi(D)[Q])\big) \leq \Phi\big(x_1(\varphi(D)[P]), \ldots, x_{n-m}(\varphi(D)[P])\big)$$

for all polynomials $P, Q \in \mathcal{H}_n$ such that $Q \preccurlyeq P$. In particular, the inequality

$$\sum_{i=1}^{n-m} f\big(x_i(\varphi(D)[Q])\big) \leq \sum_{i=1}^{n-m} f\big(x_i(\varphi(D)[P])\big)$$

holds for any convex function $f : \mathbb{R} \to \mathbb{R}$. $\hfill\square$

In the same spirit, Theorem 3 and Corollaries 7–8 combined with Theorem 1 lead to the following inequalities.

Corollary 10. *Let $n \in \mathbb{N}$ and $\varphi \in \mathcal{LP}'_0$. For any pair $(s, t) \in \mathbb{R}^2$ satisfying $|s| \leq |t|$ and $st \geq 0$ and for any Schur-convex function $\Phi : \mathbb{R}^n \to \mathbb{R}$ one has*

$$\Phi\big(x_1(\varphi(sD)[P]), \ldots, x_n(\varphi(sD)[P])\big) \leq \Phi\big(x_1(\varphi(tD)[P]), \ldots, x_n(\varphi(tD)[P])\big)$$

whenever $P \in \mathcal{H}_n$. In particular, the inequalities

$$\sum_{i=1}^{n} f\big(x_i(\varphi(sD)[P])\big) \leq \sum_{i=1}^{n} f\big(x_i(\varphi(tD)[P])\big)$$

$$\sum_{i=1}^{n} f\big(x_i(P)\big) \leq \sum_{i=1}^{n} f\big(x_i(\varphi(tD)[P])\big)$$

hold for any convex function $f : \mathbb{R} \to \mathbb{R}$.   □

Let $\mathcal{LP}''$ denote the class of entire functions of the form

$$\varphi(x) = c x^m e^{bx} \prod_{k=1}^{\infty} (1 - \alpha_k x),$$

where $c \in \mathbb{R} \setminus \{0\}$, $m \in \mathbb{Z}_+$, $b \leq 0$, $\alpha_k \geq 0$ and $\sum_{k=1}^{\infty} \alpha_k < \infty$, so that $\mathcal{LP}'' \subset \mathcal{LP}$. It is well-known that $\mathcal{LP}''$ consists precisely of those functions which are locally uniform limits in $\mathbb{C}$ of sequences of hyperbolic polynomials having only positive zeros (cf. [L, Ch. 8]). According to the terminology introduced by Pólya and Schur, a real entire function ψ is called a function of *type I* in the Laguerre-Pólya class if either $\psi(x) \in \mathcal{LP}''$ or $\psi(-x) \in \mathcal{LP}''$. For $m \in \mathbb{Z}_+$ we set $\mathcal{LP}''_m = \mathcal{LP}'' \cap \mathcal{LP}_m$. Let $P \in \mathcal{H}_n$ with $n \geq 1$ be such that $x_i(P) > 0$ for $1 \leq i \leq n$. Using Lemma 2 and polynomial approximations as in (1.13) and (1.14) one can show that if $\varphi \in \mathcal{LP}''_m$ and $n \geq m + 1$ then $x_i(\varphi(D)[P]) > 0$ for $1 \leq i \leq n - m$. These observations allow us to derive new inequalities involving differential operators associated with functions of type I in the Laguerre-Pólya class. The first two inequalities listed in Corollary 11 below correspond to the following special choices of convex functions in Corollary 9: minus the Shannon entropy $-H(x) = x \log x$ and minus the Renyi entropies $\log(\sum_{i=1}^{n} x_i^k)$ for $k \geq 1$, respectively. These are in fact easy consequences of the third inequality, which is actually the most general inequality of this type.

Corollary 11. *Let $n, m \in \mathbb{Z}_+$ with $n \geq m + 1$. For any $\varphi \in \mathcal{LP}''_m$ one has*

$$\sum_{i=1}^{n-m} x_i(\varphi(D)[Q]) \log x_i(\varphi(D)[Q]) \leq \sum_{i=1}^{n-m} x_i(\varphi(D)[P]) \log x_i(\varphi(D)[P]),$$

$$\sum_{i=1}^{n-m} [x_i(\varphi(D)[Q])]^k \leq \sum_{i=1}^{n-m} [x_i(\varphi(D)[P])]^k, \quad k \in [1, \infty),$$

$$r(r - 1) \sum_{i=1}^{n-m} [x_i(\varphi(D)[Q])]^r \leq r(r - 1) \sum_{i=1}^{n-m} [x_i(\varphi(D)[P])]^r, \quad r \in \mathbb{R},$$

for all polynomials $P, Q \in \mathcal{H}_n$ with positive zeros that satisfy $Q \preccurlyeq P$.   □

3.2. **Multiplier sequences, spectral order and isotonic operators.** It is natural to ask whether the spectral order is preserved by linear operators other than those of Laguerre-Pólya type (cf. Problem 3 below). Clearly, any such operator should necessarily map hyperbolic polynomials to hyperbolic polynomials of the same degree. An important class of operators that one may consider in this context is the class of diagonal operators (in the basis of standard monomials) that preserve hyperbolicity. This is the class of multiplier sequences of the first kind, which was completely characterized by Pólya and Schur in [PS].

Definition 5. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be an arbitrary sequence of real numbers and let $T_\Gamma \in \operatorname{End}\Pi$ be given by $T_\Gamma[x^n] = \gamma_n x^n$, $n \in \mathbb{Z}_+$. Then Γ is called a *multiplier sequence of the first kind* if T_Γ preserves the class of hyperbolic polynomials.

For convenience, we denote by $\mathcal{PS}_I$ the set of all multiplier sequences of the first kind and we let Π_n be the $(n+1)$-dimensional subspace of Π consisting of all complex polynomials of degree at most n, so that $\mathcal{H}_n \subset \Pi_n$. If $\Gamma = \{\gamma_k\}_{k=0}^{\infty} \in \mathcal{PS}_I$ and $\gamma_n \neq 0$ for some $n \in \mathbb{N}$ we define the *n-th normalized truncation* of Γ to be the finite sequence $\Gamma_n = \left\{\dfrac{\gamma_0}{\gamma_n}, \ldots, \dfrac{\gamma_{n-1}}{\gamma_n}, 1\right\}$. Obviously, Γ_n induces a well-defined linear operator $T_{\Gamma_n} \in \operatorname{End}\Pi_n$ that satisfies $T_{\Gamma_n}(\mathcal{H}_n) \subseteq \mathcal{H}_n$.

Problem 1. Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty} \in \mathcal{PS}_I$ be such that $\gamma_n \neq 0$, $n \in \mathbb{N}$. Is it true that for any $n \in \mathbb{N}$ the operator $T_{\Gamma_n} \in \operatorname{End}\Pi_n$ preserves the partial ordering $\preccurlyeq$ on $\mathcal{H}_n$, where Γ_n is the n-th normalized truncation of Γ?

The condition $\gamma_n \neq 0$, $n \in \mathbb{N}$, imposed in Problem 1 is far from being as restrictive as it may first appear and is actually quite natural in view of well-known properties of multiplier sequences of the first kind (see, e.g., [L]). Indeed, if $\Gamma = \{\gamma_k\}_{k=0}^{\infty} \in \mathcal{PS}_I$ then $\{\gamma_{i+k}\}_{k=0}^{\infty} \in \mathcal{PS}_I$ for any $i \in \mathbb{N}$. Moreover, if $\gamma_0 \neq 0$ and $\gamma_i = 0$ for some $i \in \mathbb{N}$ then $\gamma_j = 0$ for all $j \geq i$. It follows that either Γ contains only zero terms except for a finite number of consecutive nonzero elements or there exists $i \in \mathbb{Z}_+$ such that $\gamma_k = 0$ for $k \leq i - 1$ and $\gamma_k \neq 0$ if $k \geq i$.

As an example, consider the sequence $\Gamma = \{k\}_{k=0}^{\infty}$ consisting of the Maclaurin coefficients of xe^x. Clearly, $T_\Gamma[P(x)] = xP'(x)$ for any $P \in \Pi$ hence $T_{\Gamma_n}(\mathcal{H}_n) \subseteq \mathcal{H}_n$, $n \in \mathbb{N}$. Note that in this case Lemma 5 and Theorem 1 imply that T_{Γ_n} preserves indeed all the poset structures $(\mathcal{H}_n, \preccurlyeq)$, $n \in \mathbb{N}$. Similar considerations show that the answer to Problem 1 is affirmative for multiplier sequences of the following type.

Proposition 6. *Let $m \in \mathbb{N}$, $p \in \mathbb{Z}_+$ and consider the sequence $\Gamma = \{H(k+p)\}_{k=0}^{\infty}$, where $H(x) = \prod_{i=0}^{m-1}(x - i)$. Then $\Gamma \in \mathcal{PS}_I$ and for any $n \geq \max(1, m - p)$ the operator T_{Γ_n} preserves the partial ordering $\preccurlyeq$ on $\mathcal{H}_n$.*

Proof. If $n \in \mathbb{N}$ and $P(x) = \sum_{k=0}^{n} x^k \in \Pi_n$ then

$$T_\Gamma[P(x)] = \sum_{k=0}^{n} H(k + p)a_k x^k = x^{m-p}[x^p P(x)]^{(m)}$$

and so by Rolle's theorem Γ is a multiplier sequence of the first kind. The same arguments further show that $T_{\Gamma_n}(\mathcal{H}_n) \subseteq \mathcal{H}_n$ for all $n \geq \max(1, m - p)$ since $H(n + p) \neq 0$ for such n. Using Lemma 5 and Theorem 1 (i) one can easily check that $x^{m-p}[x^p Q(x)]^{(m)} \preccurlyeq x^{m-p}[x^p P(x)]^{(m)}$ whenever $n \geq \max(1, m - p)$ and $P, Q \in \mathcal{H}_n$ are such that $Q \preccurlyeq P$. $\square$

A somewhat different version of Problem 1 is as follows.

Problem 2. Fix $n \in \mathbb{N}$ and consider a finite sequence $\Lambda = \{\lambda_k\}_{k=0}^{n}$ with associated operator $T_{\Lambda_n} \in \operatorname{End}\Pi$ given by $T_{\Lambda_n}[x^k] = \lambda_k x^k$, $0 \leq k \leq n$, $T_{\Lambda_n}[x^k] = 0$, $k > n$. If $\lambda_n = 1$ and $T_{\Lambda_n}(\mathcal{H}_n) \subseteq \mathcal{H}_n$ is it true that T_{Λ_n} preserves the spectral order on $\mathcal{H}_n$?

The answer to Problem 2 is trivially affirmative if $n = 1$ and elementary computations show that this holds for $n = 2$ as well. Indeed, if $\Lambda = \{\lambda_0, \lambda_1, 1\}$ is a sequence that satisfies the above hypotheses then $\lambda_0 \geq 0$ since $T_{\Lambda_n}[x^2 - 1] \in \mathcal{H}_2$. Given two polynomials $P(x) = x^2 + ax + b \in \mathcal{H}_2$ and $Q(x) = x^2 + cx + d \in \mathcal{H}_2$ with $Q \preccurlyeq P$ one has $a = c$, $a^2 \geq 4\max(b, d)$ and $\sqrt{a^2 - 4d} \leq \sqrt{a^2 - 4b}$. From $\lambda_0 \geq 0$ we get $\sqrt{\lambda_1^2 a^2 - 4\lambda_0 d} \leq \sqrt{\lambda_1^2 a^2 - 4\lambda_0 b}$, which shows that $T_{\Lambda_n}[Q] \preccurlyeq T_{\Lambda_n}[P]$.

Problem 2 may actually be viewed as a special case of a yet more general problem. Fix $n \in \mathbb{N}$ and recall the monoid $\mathcal{A}_n$ defined in (1.16). Let $\mathcal{A}_n^{\preccurlyeq}$ denote the submonoid of $\mathcal{A}_n$ consisting of all operators that preserve the poset structure $(\mathcal{H}_n, \preccurlyeq)$, that is,

$$\mathcal{A}_n^{\preccurlyeq} = \{T \in \mathcal{A}_n \mid T[Q] \preccurlyeq T[P] \text{ if } P, Q \in \mathcal{H}_n, \, Q \preccurlyeq P\}.$$

Recall also the submanifold $\mathcal{H}_n^0$ of $\mathcal{H}_n$ from (1.18) and consider the submonoid $\mathcal{A}_n^0$ of $\mathcal{A}_n$ given by

$$\mathcal{A}_n^0 = \{T \in \mathcal{A}_n \mid T\big(\mathcal{H}_n^0\big) \subseteq \mathcal{H}_n^0\}.$$

Problem 3. Describe all operators in $\mathcal{A}_n^{\preccurlyeq}$. Is it true that $\mathcal{A}_n^{\preccurlyeq} = \mathcal{A}_n^0$ for all $n \in \mathbb{N}$?

Conjecture 1. *Problems 1–3 have all affirmative answers.*

Remark 7. The linear transformations on $\mathbb{R}^n$ that preserve the majorization relation $\prec$ between n-vectors of real numbers were characterized in [An2, DV].

Note that Problem 3 implicitly addresses and further motivates both the question of describing all operators in the monoid $\mathcal{A}_n$ itself (cf. [B, Problem 2 (iii)]) and its version with no restriction on the degrees that may be formulated as follows.

Problem 4. Characterize all operators in the monoid $\tilde{\mathcal{A}} := \{T \in \operatorname{End} \Pi \mid T(\mathcal{H}) \subseteq \mathcal{H}\}$, where $\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_n$.

Problem 4 is actually a long-standing open problem of fundamental interest in the theory of distribution of zeros of polynomials and transcendental entire functions (see [CC1, Problem 1.3]). Significant progress towards a complete solution to Problem 4 was recently made in [BBS].

The above results and those of [B, BP, BS] show that even a partial knowledge of the operators in $\mathcal{A}_n$ leads to new interesting information on the relative geometry of the zeros of a hyperbolic polynomial and the zeros of its images under such operators. Several related questions arise naturally in this context. For instance, Problem 2 (ii) in [B] asks whether it is possible to describe the spectral order by means of the action of linear (differential) operators on the partially ordered manifold $(\mathcal{H}_n, \preccurlyeq)$. This would provide a new characterization of classical majorization which in a way would be dual to the usual characterization in terms of doubly stochastic matrices given in Theorem 1.

It would also be interesting to know whether there are any "infinite-dimensional" analogs of Theorems 2 and 3. Indeed, it is well known that the class $\mathcal{LP}$ is closed under differentiation [L]. A more general closure property was established in [CC2], where various types of infinite order differential operators acting on $\mathcal{LP}$ were studied in detail. In particular, Lemmas 3.1 and 3.2 in *loc. cit.* show that the subset of $\mathcal{LP}$ consisting of entire functions of genus 0 or 1 is stable under the action of differential operators of Laguerre-Pólya type. Moreover, there are several known extensions of classical majorization to infinite sequences of real numbers [MO, p. 16]. One may therefore ask if these extensions or some appropriate modifications could lead to generalizations of the above results to differential operators acting on transcendental entire functions in the class $\mathcal{LP}$.

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